1 Basic solving with Cholesky

Solving a linear least-squares system:

$$\arg\min_x \|Ax - b\|^2$$

Set derivative equal to zero:

$$0 = 2A^T (Ax - b)$$
$$0 = A^T Ax - A^T b$$

For comparison, with QR we do

$$0 = R^T Q^T QRx - R^T Qb$$
$$= R^T Rx - R^T Qb$$
$$Rx = Qb$$
$$x = R^{-1}Qb$$

But with Cholesky we do

$$0 = R^T R R^T Rx - R^T Rb$$
$$= R^T Rx - b$$
$$= Rx - R^{-T}b$$
$$x = R^{-1}R^{-T}b$$

2 Frontal (rank-deficient) solving with Cholesky

To do multi-frontal elimination, we decompose into rank-deficient conditionals.

$$\begin{bmatrix} . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \end{bmatrix} \rightarrow$$

$$\begin{bmatrix} R^T & 0 \\ S^T & C^T \end{bmatrix} \begin{bmatrix} R & S \\ 0 & C \end{bmatrix} = \begin{bmatrix} F^T F & F^T G \\ G^T F & G^T G \end{bmatrix}$$

$$R^T R = F^T F$$

$$R^T S = F^T G$$
$$S = R^{-T} F^T G$$

$$S^T S + C^T C = G^T G$$
$$G^T F R^{-1} R^{-T} F^T G + C^T C = G^T G$$
$$G^T Q R R^{-1} R^{-T} R^T Q^T G + C^T C = G^T G$$
$$\textbf{if } R \textbf{ is invertible, } G^T G + C^T C = G^T G$$
$$C^T C = 0$$